Question A – Recursion and Induction – 20 Marks

Given the sequence a_n defined with the recurrence relation:

$a_0 = 1$

$a_n = n\,(a_{n-1})^2$ for $n \geq 1$

Terms of a Sequence (5 marks)

$a_1 = 1 \,.\, 1^2 = 1$
$a_2 = 2 \,.\, 1^2 = 2$
$a_3 = 3 \,.\, (2 \,.\, 1^2)^2 = 3 \,.\, 2^2 \,.\, 1^4 = 12$
$a_4 = 4 \,.\, (3 \,.\, 2^2 \,.\, 1^4)^2 = 4 \,.\, 3^2 \,.\, 2^4 \,.\, 1^8 = 576$
$a_5 = 5.\, (4 \,.\, 3^2 \,.\, 2^4 \,.\, 1^8)^2 = 5 \,.\, 4^2 \,.\, 3^4 \,.\, 2^8 \,.\, 1^{16} = 1,658,880$

Iteration (3 marks)

$$a_n = \prod_{i=1}^{n} i^{2^{n-i}} = \prod_{i=0}^{n-1} (n-i)^{2^i}$$

Proof by induction (12 marks)

Show that $2 \mid n^2 - n$ for all positive integers n by weak induction. No other method is acceptable.

Define the predicate P(n) to be $2 \mid n^2 - n$.

We are going to show that $\forall n \in \mathbb{N}^+ \, P(n)$

Proof:
Base case:
When n=1, $n^2 - n = 1-1 = 0 = 2 \,.\, 0$, so $2 \mid n^2 - n$
i.e. P(1) is true.
Inductive Step:
Assume that P(n) is true for some n in $\mathbb{N}^+$

This means that $\exists k \in \mathbb{Z},\ n^2 - n = 2k$

Show that P(n+1) is true.
$(n+1)^2 - (n+1) = n^2 + 2n + 1 - n - 1 = n^2 - n + 2n$ by algebra
$= 2k+2n$ by inductive hypothesis
$= 2(k+n)$ by algebra

Since k,n are integers and $\mathbb{Z}$ is closed under +, then $k+n \in \mathbb{Z}$

Therefore $2 \mid (n+1)^2 - (n+1)$
QED by induction

Question B – Number Theory – 20 marks

Euclidian Algorithm (5 marks)

gcd (598, 1287) = gcd(1287,598)
= gcd(598,1287 mod 598) = gcd(598,91)
= gcd(91,598 mod 91) = gcd(91,52)
= gcd(52,91 mod 52) = gcd(52,39)
= gcd(39,52 mod 39) = gcd(39,13)
= gcd(13,39 mod 13) = gcd(13,0) = 13

Mod Proof (15 marks)

Prove that for any integers A, B, a, d such that $d \neq 0$,
if A mod d = a and B mod d = 1 then AB mod d = a

Proof:

Let A, B, a, d be integers such that $d \neq 0$ and A mod d = a and B mod d = 1

We'll show that AB mod d = a

Since A mod d = a then by the QRT: A = (A div d) . d + a (1) and $0 \leq a < d$ (2)

Since B mod d = 1 then by the QRT: B = (B div d) . d + 1 (3)

So AB = ((A div d) . d + a) . ((B div d) . d + 1) by substituting with (1) and (3)
= [(A div d)(B div d).d + (B div d).a + (A div d)] . d + a

Let p = [(A div d)(B div d).d + (B div d).a + (A div d)]

Then AB = p.d + a (4)

Since all the terms in p are integers and $\mathbb{Z}$ is closed under div, + and . , then $p \in \mathbb{Z}$. (5)

By the QRT: $\exists! q, r \in \mathbb{Z}$ AB = q.d + r and $0 \leq$ r = AB mod d < d

Therefore, since by (4): AB = p.d + a and by (2): $0 \leq a < d$, where $p, a \in \mathbb{Z}$ by (5)

then a must be AB mod d (because of the uniqueness part of the QRT)

QED

Question C – Graph Theory – 20 marks

Graph Degrees (12 marks)

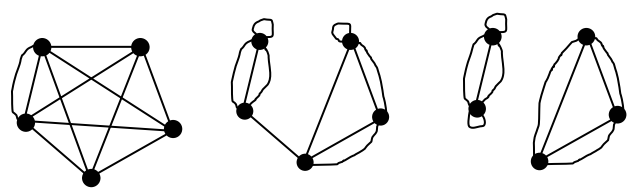
For each of the following questions, either draw a graph with the requested properties, or explain **convincingly** (possibly by quoting a theorem) why such a graph cannot be drawn.

a) A graph with 5 vertices of degrees 5, 5, 4, 4, 3

This graph cannot be drawn because the degree of a graph (sum of degree of vertices) must be even, and in this case the sum of the degree of these 5 vertices is 21 which is odd.

b) A graph with 5 vertices of degrees 5, 5, 4, 4, 4

There are many possible answers. Here are 3:



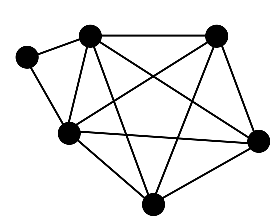
c) A simple graph with 5 vertices of degrees 5, 5, 4, 4, 4

Simple graphs do not contain parallel edges or loops, so the maximum degree of any edge in a simple graph with n vertices is n-1 as each vertex has at most n-1 incident edges, one for each of the other vertices.

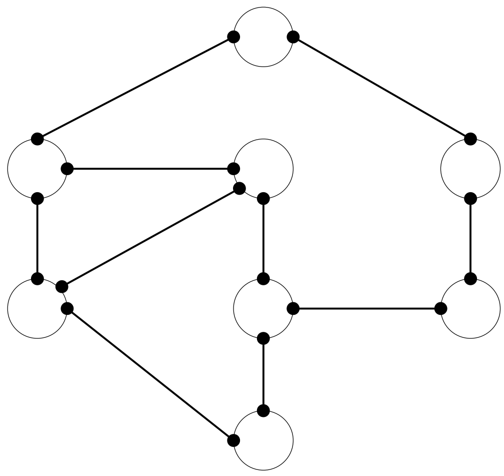
Therefore it is not possible to draw a simple graph with n vertices, at least one of which has degree n,

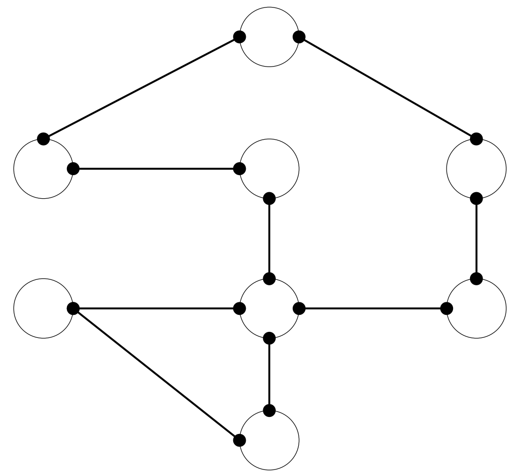
d) A simple graph with at least 5 vertices which have degrees 5, 5, 4, 4, 4. The other vertices have whichever degree seems appropriate.

Again there are many possible answers. One simple answer is built by adding one vertex to K_5 and connecting it to 2 of K_5's vertices, thus bringing their degrees up from 4 to 5:



Circuits (8 marks)





a) Find an Euler circuit in G that starts at v_1.

A graph contains an Euler circuit iff it is connected and every vertex has an even degree. G, however, has some vertices with odd degree (v_2, v_3, v_5, v_6). Therefore it cannot contain an Euler circuit.

b) Find a Hamiltonian circuit in G that starts at v_1

There are 2 Hamiltonian circuits:

- $v_1e_1v_2e_3v_3e_5v_5e_{10}v_8e_8v_6e_9v_7e_7v_4e_2v_1$
- $v_1e_2v_4e_7v_7e_9v_6e_8v_8e_{10}v_5e_5v_3e_3v_2e_1v_1$

c) Find an Euler circuit in H that starts at v_1

There are 4 Euler circuits:

- $v_1e_2v_4e_5v_7e_7v_6e_6v_5e_8v_8e_9v_6e_4v_3e_3v_2e_1v_1$
- $v_1e_2v_4e_5v_7e_7v_6e_9v_8e_8v_5e_6v_6e_4v_3e_3v_2e_1v_1$
- $v_1e_1v_2e_3v_3e_4v_6e_6v_5e_8v_8e_9v_6e_7v_7e_5v_4e_2v_1$
- $v_1e_1v_2e_3v_3e_4v_6e_9v_8e_8v_5e_6v_6e_7v_7e_5v_4e_2v_1$

d) Find a Hamiltonian circuit in H that starts at v_1

Theorem: If H contained a Hamiltonian circuit, then it would have a connected subgraph which contained all its vertices, all of which would be of degree 2.

To get such a subgraph, all the edges incidents on vertices in V(H)-{v_6} would have to be preserved because all the vertices in that set have degree 2. So the subgraph would have to include edges e_4,e_6,e_7,e_9 and as a result vertex v_6 would have to be of degree 4 which contradicts the theorem.

v1
e1
e3
v2
v3
e4
e
e5
v5
v6
e
e10
v8